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Soliton Generation and Nonlinear Wave Propagation [and Discussion]

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Soliton generation and nonlinear wave propagation

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Soliton generation by various means is described. First, experimental results of J. V. Wehausen and coworkers on solitons generated by a ship model in a towing tank are presented. Then T. Y. Wu's related Boussinesq system of equations for shallow water motion with a moving pressure disturbance on the free surface is introduced. Numerical solutions of this system by D. M. Wu and T. Y. Wu are shown to compare well with the experimental results. Similar numerical results on an initial-boundary value problem for the K.d.V. equation by C. K. Chu and coworkers are presented, which also yield soliton generation. Then J. P. Keener and J. Rinzel's analysis of pulse generation in the Fitzhugh–Naqumo model of nerve conduction is described. Next, G. B. Whitham's modulation theory of nonlinear wave propagation is explained and the problem of relating its results to initial and boundary data is mentioned. Asymptotic methods for solving this problem for the K.d.V. equation are described. They include the Lax–Levermore theory for the case of small dispersion, its extension by S. Venakides, and the centered simple wave solution of the modulation equations by A. V. Gurevitch and L. P. Pitaevskii. Finally, the theory of weakly nonlinear waves of Choquet–Bruhat and of J. K. Hunter and the present author is described.

1. Introduction

The nonlinear partial differential equations that govern wave phenomena have resisted analysis for many years. During and just after World War II some progress was made on the equations of gas dynamics, by exploiting the theory of characteristics and applying it to shock waves (Courant & Friedrichs 1948). In addition, computers have become available, and they have been used very successfully to solve many wave problems. Since then there have been two more major advances: the development of modulation theory by Whitham (1965) and the development of the inverse scattering transform method for solving the Korteweg–de Vries (K.d.V.) equation by Gardner *et al.* (1967).

These four methods – characteristics, computers, modulation theory and the inverse scattering transform – have provided the basis for most of the recent work on nonlinear waves. Nevertheless all these methods are limited, so there is need for further development of them, as well as for the development of other methods.

In the theory of linear wave propagation, short-wave asymptotic analysis has been particularly successful. It has led to ray methods, such as the geometrical theory of diffraction, which have been used very widely. Asymptotic analysis has also been applied to nonlinear wave propagation. For example, modulation theory has been derived by this means, and the derivation reveals that the theory is in fact nonlinear geometrical optics. In addition, the K.d.V. equation has been derived asymptotically from many other more complicated equations, or systems, as the generic equation governing dispersive waves, just as the Burgers equation is generic for dissipative waves. Furthermore, asymptotics has been used to solve the K.d.V. equation in various ways.

I shall attempt to review some of the recent progress in the theory of nonlinear wave propagation, emphasizing soliton generation, modulation theory and asymptotic analysis.

2. Soliton generation in a towing tank

A ship design is always tested by making a small-scale wax model of the ship hull, suspending the model from a carriage so that it is submerged to the proper depth in the water in a towing tank, and then towing the carriage along a track. In this way the ship model moves through the water at the towing speed and creates a wave pattern behind it. These waves carry energy away from the model and thus lead to the wave-making resistance of the hull. The pattern rapidly achieves a permanent form when viewed from the carriage.

Recently, J. V. Wehausen and his collaborators at Berkeley reported some unusual wave patterns that they observed in a towing tank filled to the shallow depth of 0.5 feet (15 cm) (Huang *et al.* 1982). As the wave pattern began to form it spread out into a single wave across the tank, detached itself from the ship and then travelled ahead of the ship as a solitary wave. Then another such wave was generated, and so on. This occurred at values of the Froude number $Fr = U/\sqrt{gh}$ from about 0.75 to about 1.3. Here U is the ship model speed, g is the acceleration of gravity and h is the depth of water. For $Fr < 1$ the successive solitons decreased in amplitude, but not for $Fr > 1$. At all values of Fr, a shelf developed ahead of the model. For $Fr \geqslant 1.3$, when no solitons are formed, a bore develops ahead of the model and travels with it. This agrees with the fact that the maximum velocity of an isolated solition is $c_{\max} = 1.29\sqrt{gh}$, so for $Fr \geqslant 1.3$ a soliton could not get away from the model.

Measurements of the period T of soliton generation showed that the dimensionless period UT/h is linear in the Froude number for $0.8 \leqslant Fr \leqslant 1.1$. In addition, the dimensionless soliton velocity $c/\sqrt{gh}$ varied linearly with Fr for $0.8 \leqslant Fr \leqslant 1.2$. Furthermore the resistance measured on the model varied periodically in time, reaching a maximum at about the time a soliton left the model.

These experimental results have been compared with the results of numerical calculations on the following pair of equations of Boussinesq type derived by Wu (1981):

$$\left.\begin{aligned} &\eta_t + (h+\eta)\,u_x + \eta_x\,u = 0,\\ &u_t + uu_x + g\eta_x - \tfrac{1}{3}h^2 u_{xxt} = -p_{0x}/\rho. \end{aligned}\right\} \tag{1}$$

Here $y = \eta(x,t)$ is the equation of the free surface, $y = -h$ is the equation of the bottom, $u(x,t)$ in the depth-averaged horizontal velocity of the fluid, ρ is the fluid density, and $p_0(x,t)$ is the pressure applied to the free surface. The calculations were done by Wu & Wu (1982) at CalTech. They used the moving pressure distribution

$$\left.\begin{aligned} p_0(x,t) &= p_{0m}\,\frac{1}{2}\left[1-\cos\frac{2\pi}{L}\,(x+Ut)\right], \quad 0 < x+Ut < L,\\ &= 0, \quad \text{elsewhere.} \end{aligned}\right\} \tag{2}$$

As initial conditions they took

$$\eta(x,0) = -p_0(x,0)/\rho g,\; u(x,0) = 0. \tag{3}$$

The results of the calculations were similar to the experimental results. Solitons were generated periodically at $Fr = 0.9$, 1.0 and 1.1, but not at $Fr = 1.175$, nor for larger Fr. In

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both the numerical and experimental results, behind the solitons there was a long region within which the surface was smooth, followed by a region of waves oscillating about the mean surface level. These calculations were repeated by R. C. Ertekin & J. V. Wehausen (unpublished work on ship-generated solitons done at Department of Naval Architecture, University of California, Berkeley (1983)) with similar results. In addition they solved the same problem by using equations obtained from the theory of fluid sheets developed by Green & Naghdi (1976, 1977). The two sets of results agreed with one another and with Wu's results, for small disturbances. However, for larger disturbances the Green–Naghdi equations gave reasonable results, while Wu's equations gave results that behaved wildly. This is presumably a consequence of the fact that the Green–Naghdi equations conserve mass, momentum and energy exactly, while Wu's equations do not.

3. Soliton generation in initial-boundary value problems

The mathematical problems mentioned in §2 were initial value problems on the entire real line with inhomogeneous terms in the equation. However, the physical problem of wave generation by a moving model might also be formulated as a piston problem, which gives rise to an initial-boundary value problem on the half line. Therefore, it is of interest to consider some results for such problems.

Chu *et al.* (1983) have considered the following initial boundary value problem for the K.d.V. equation:

$$u_t + uu_x + u_{xxx} = 0, \quad t > 0, \quad x > 0, \tag{4}$$

$$u(x, 0) = 0, \tag{5}$$

$$u(0, t) = u_0(t). \tag{6}$$

For $u_0(t)$ they chose a trapezoidal pulse, i.e. one that increased linearly for a short time, remained at the constant value u_0 for a long time, and then decreased linearly to zero in a short time. They solved this problem numerically, by finite differences, for two different durations of the period of constancy. In both cases solitons were produced: two for the shorter duration pulse and three for the longer pulse. The initial solitons were nearly identical in the two cases. Their speeds quickly approached the value $\frac{2}{3}u_0$, and their amplitudes approached $2u_0$. For the longer pulse the second soliton was nearly the same as the first, but in both cases the final soliton was smaller.

It seems clear from the results that the longer the pulse the more solitons it will produce, and that they will all tend to the amplitude $2u_0$ and speed $\frac{2}{3}u_0$, except for the final one. Chu *et al.* also calculated solitons generated for the K.d.V.–Burgers equation, which contains the extra term $-\nu u_{xx}$.

Bona & Winther (1983) have proved existence and uniqueness theorems for the initial-boundary value problem for the K.d.V. equation. However, there are no analytical methods available for constructing such solutions, and obtaining them is a worthwhile goal. Therefore it is of interest to consider the Fitzhugh–Nagumo equation of nerve conduction, for which there are results due to Keener & Rinzel (1983). That equation, for the potential $u(x, t)$ across the nerve membrane, is

$$\left.\begin{aligned} u_t &= u_{xx} + u(1-u)(u-a) - v, \\ v_t &= \epsilon(u - \gamma v). \end{aligned}\right\} \tag{7}$$

Here a, ϵ and γ are constants with $0 < a < \frac{1}{2}$, $\epsilon > 0$ and $\gamma > 0$, and v is a recovery current. The signalling problem involves solving this equation for $x > 0$ and $t > 0$ with the initial value zero and with a prescribed current $\frac{1}{2}I$ at the input $x = 0$:

$$u_x(0, t) = -\tfrac{1}{2}I. \tag{8}$$

It is known experimentally that nerves can transmit isolated pulses, and a train of them is produced if I exceeds a threshold value I_0. However, none are produced if $I < I_0$. It is known analytically that the Fitzhugh–Nagumo equation has pulse solutions and periodic travelling wave solutions. It is found numerically that the initial-boundary value problem formulated above agrees with experiments in yielding a train of pulses if $I > I_0$ and not otherwise.

Keener & Rinzel (1983) observed that for any $I > 0$ the boundary value problem above has a time-independent solution $U(x)$, which tends to zero at $x = \infty$. They examined the linear stability of that solution and showed that it is stable for $I < I_c$ and unstable for $I > I_c$, where I_c is a certain positive constant. At I_c the solution undergoes a Hopf bifurcation, with the emergence of a time-periodic solution of the linearized problem. The value of I_c is close to the numerically determined value of the threshold I_0. This suggests an explanation of the results observed experimentally and numerically.

To apply this analysis to the K.d.V. equation (4), we note that it has the following time-independent solution, which decays at infinity:

$$U(x) = -[x/2\sqrt{3} + (-U_0)^{-\frac{1}{2}}]^{-2}, \quad x > 0. \tag{9}$$

Here U_0 is the value of U at $x = 0$, and (9) holds only with $U_0 < 0$. The variational equation governing its linear stability is

$$v_t + Uv_x + vU_x + v_{xxx} = 0, \quad x > 0. \tag{10}$$

The variables separate in (10), so with $v(x, t) = \mathrm{e}^{\lambda t}\, w(x)$ the stability analysis reduces to the study of the eigenvalues λ of the ordinary differential equation

$$\lambda w + Uw_x + wU_x + w_{xxx} = 0, \quad x > 0. \tag{11}$$

The solution w must vanish at $x = 0$ and at $x = \infty$. This problem does not seem to have been investigated.

There is no evidence that there is a positive threshold for the K.d.V. equation, and some results indicate that there is none. In particular, the fact that (9) holds only for $U_0 < 0$, and that there is no bounded time-independent solution for $U_0 > 0$, shows that then $u(x, t)$ cannot become time-independent for t large. Thus there probably is soliton generation for all positive boundary values U_0. We shall return to this problem later from a different point of view.

Finally we consider the problem of the motion produced by a piston moving with the constant velocity $c > 0$. Then we seek a solution of (4) with the velocity u equal to c at the piston, i.e. $u = c$ at $x = ct$, and $u = 0$ at $x = +\infty$. We seek a solution of the form $u = u(x-ct)$ and we find that u is determined by the equation

$$x - ct = \left(\frac{3}{c}\right)^{\frac{1}{2}} \int_0^{1-u/c} \frac{\mathrm{d}w}{(w-1)\,(w+2)^{\frac{1}{2}}}.$$

This solution represents a steadily moving disturbance that decreases monotonically with distance from the piston. It could represent the bore ahead of a moving ship in a canal. When

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the ship stops moving, this disturbance might propagate onward as a solitary wave, such as the one observed by Scott Russell (1844). A similar solution for the Boussinesq system (1) with $p_0 = 0$ is

$$\frac{x-ct}{h} = \int_0^{1-u/c} \left[\left(w^3-1+\frac{6gh}{c^2}+3\right)(1-w)+\frac{6gh}{c^2}\ln w\right]^{-\frac{1}{2}} dw.$$

4. Modulation theory of nonlinear wave propagation

Suppose some partial differential equation, or system of such equations, has constant coefficients. Then it may possess travelling plane wave solutions $U(\phi)$, which are functions of one variable $\phi = \boldsymbol{k}\cdot\boldsymbol{x}-\omega t$. Here $\boldsymbol{k}$ is the wavevector, which points in the direction of propagation, and ω is the angular frequency of the wave. The partial differential equation reduces to an ordinary differential equation for $U(\phi)$. The periodic solutions of this equation yield periodic travelling plane waves. When the equation for U is of second order, the solution has two constants of integration. One is an additive phase shift η because the equation has constant coefficients, and the other is a measure of the size of U and is called an amplitude a. Thus the general solution is of the form $U(\phi+\eta, a)$.

Whitham (1965) devised a theory to determine how such waves propagate when they are not necessarily plane, when their frequency and amplitude vary, and when the coefficients in the equation are not constant. He sought a solution of the form $u(x, t) = U[\phi(x, t), a(x, t)]$ and substituted it into the Lagrangian variational principle, which was assumed to govern the system. Then he averaged the Lagrangian over a period of U. Upon varying this averaged Lagrangian with respect to ϕ and a he derived a system of coupled partial differential equations for these quantities. These are the modulation equations that describe the propagation of slowly varying waves.

Subsequently Whitham's student Luke (1966) showed that for a particular nonlinear wave equation, the modulation equations could be derived by a systematic asymptotic expansion in a small parameter. The parameter could be identified as the ratio of the wavelength to a typical scale length associated with the variation of the medium or of the amplitude. Then Kogelman & Keller (1973) showed how to do the same for a large class of equations, and to obtain higher order terms. Later Flaschka *et al.* (1980) derived modulation equations for interacting wave solutions of the K.d.V. equation.

In view of the success of modulation theory, there are various directions in which it should be extended to make it more useful. I give three here.

(*a*) The theory does not describe how waves arise in initial and initial-boundary value problems. Some initial and boundary layer solutions are needed to match the solution given by modulation theory to the initial and boundary data.

(*b*) Methods for solving the equations of modulation theory are needed, since those equations themselves are a system of nonlinear partial differential equations.

(*c*) Theories of reflection, refraction, diffraction and scattering of nonlinear waves are needed to extend the range of modulation theory to such problems.

Progress has been made on all of these topics, and I shall now describe some of it.

5. Asymptotic solutions of the K.d.V. equation

Concerning (*a*), the main results have been achieved for the initial value problem for the K.d.V. equation by using the inverse scattering transform. The results show that as t becomes large, the solution develops into separated solitons or travelling waves (Tanaka 1973), which could then be followed further by modulation theory. However, modulation theory has not been used in that case because the explicit behaviour of the waves is obtained from the exact solution.

Lax & Levermore (1983) have studied the following form of the K.d.V. equation:

$$u_t - 6uu_x + \epsilon^2 u_{xx} = 0. \tag{12}$$

They have evaluated the weak limit of the solution asymptotically as $\epsilon \to 0$ for non-positive initial data. Venakides (1984) has extended their analysis to include positive data. The results show that for a certain initial interval, $0 < t < t_b$, the solution $u(x, t, \epsilon)$ of (12) tends to the solution $v(x, t)$ of

$$v_t - 6vv_x = 0, \quad v(x, 0) = u(x, 0). \tag{13}$$

The time t_b is the time at which the solution of (13) 'breaks', i.e. first develops a singularity. After that time waves are formed around the places where breaking occurs. The resulting multiphase waves are described by parameters satisfying the modulation equations of Flaschka *et al.* (1980). The fact that these waves have the waveform required by modulation theory has been shown by Venakides (1984). The substitution $U(x, t) = w(x/\epsilon, t/\epsilon)$ converts (12) into the same equation for w with $\epsilon = 1$. This shows that the preceding analysis applies to short waves, for which modulation theory should hold. In fact, Gurevitch & Pitaevskii (1974) used modulation theory to determine the waves formed near the time and place of breaking.

The preceding theory suggests a way to treat the initial-boundary value problem for the K.d.V. equation (12). We assume that near the boundary as well as near the initial line, we can drop the ϵ^2 term from (12) to get

$$v_t - 6vv_x = 0, \quad v(x, 0) = u(x, 0), \quad v(0, t) = u(0, t). \tag{14}$$

We can solve (14) for $v(x, t)$ by the method of characteristics until breaking occurs. Then by adapting the analysis of Lax & Levermore, and of Venakides, we can determine the initial behaviour of the waves that are produced near each breaking point. Their subsequent propagation can be followed by the single phase or multiphase modulation equations.

Another asymptotic result for the K.d.V. equation is the determination of the long-time behaviour of the solution of the initial value problem for step-like initial data. Hruslov (1976) and Venakides (1984) considered (12) with $\epsilon = 1$ and initial data that tended rapidly enough to -1 at $x = -\infty$ and to 0 at $x = +\infty$. They showed that as $t \to \infty$ there are of the order of $t^{\frac{1}{4}-\delta}$ solitons produced. More precisely, Venakides found that for $x \geqslant 4t - \gamma(t)$ where $\gamma(t) \ll t^{\frac{1}{4}}$,

$$u(x, t) \sim -\sum_{N=1}^{|\gamma|} 2 \operatorname{sech}^2 \left[x - 4t + \left(N + \frac{\alpha}{2} + \tfrac{1}{2} \right) \ln t + b_N \right]. \tag{15}$$

Here α is a constant independent of N, and the b_N are constants. This result shows how each soliton is effected by the others.

For the same kind of data, assumed to be monotonic in x, Lax & Levermore (1983) obtained

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more complete results for the long-time behaviour of $\bar{u}(x,t)$, the weak limit of the solution of (12) as $\epsilon \to 0$. They showed that as $t \to \infty$,

$$\left.\begin{aligned} \tilde{u}(x,t) &\sim -1, \quad x < -6t, \\ &\sim S(x/t), \quad -6t < x < 4t, \\ &\sim 0, \quad 4t < x. \end{aligned}\right\} \tag{16}$$

The function $S(x/t)$ is determined in terms of complete elliptic integrals. They obtained similar results for other kinds of initial data.

All the results mentioned in this section show how the waves described by modulation theory arise from the initial data. They all depend upon the exact solution provided by the inverse scattering transform. We shall now describe a case in which the waves can be obtained from the initial data by modulation theory itself.

6. Simple wave solutions of the modulation equations

For the K.d.V. equation, Whitham's (1965) modulation equations are hyperbolic. Gurevitch & Pitaevskii (1974) observed that they have centered simple wave solutions. These solutions can be used to solve the initial value problem for the modulation equations with the initial data for $u(x,t)$ being a step,

$$\left.\begin{aligned} u(x,0) &= -1, \quad x > 0, \\ &= 0, \quad x < 0. \end{aligned}\right\} \tag{17}$$

The corresponding solution of the modulation equations for the amplitude a, the wavenumber $k = \phi_x$ and the mean height $\bar{u}$ is found to consist of functions of x/t alone. Suppose the original equation is written as

$$u_t + uu_x + u_{xxx} = 0. \tag{18}$$

Then the solution is found to satisfy

$$\left.\begin{aligned} u(x,t) &\sim 0, \quad x/t < -2 \\ u(x,t) &\sim -1, \quad -\tfrac{1}{3} < x/t. \end{aligned}\right\} \tag{19}$$

In the interval $-2 < x/t < -\frac{1}{3}$, u is expressed in terms of complete elliptic integrals.

There is no reason to believe that the modulation equations should hold near the discontinuity in the initial data. However, they need not hold there for the preceding analysis to apply. As long as the solution is of the centered simple wave form for x and t large, that method of solution determines it. Fornberg & Whitham (1978) solved the problem (17) and (18) numerically and found that the solution (19), which they evaluated, described very well the main features of the numerical results.

It has apparently not been noticed that the preceding method can be used to solve certain initial-boundary value problems for the K.d.V. equation (18). For example, suppose the initial and boundary data are

$$u(x,0) = 0, \quad x < 0, \tag{20}$$

$$u(0,t) = -1, \quad t > 0. \tag{21}$$

The solution of the modulation equations for this case is a centered simple wave. It is exactly the same as for the initial value problem (17) and (18). Thus $u(x, t)$ is given by (19) for $x \leqslant 0$.

7. Weakly nonlinear waves

One way to overcome the difficulty of solving the modulation equations is to consider weakly nonlinear waves. A method for doing that, which goes beyond linear geometrical optics, was developed by Hunter & Keller (1983) following the procedure devised by Choquet-Bruhat (1969). The basic idea is to study waves in which the wave amplitude and the wavelength are both small of the same order ϵ, so that the solution is roughly of the form $\epsilon u(x/\epsilon)$. Then $\partial_x[\epsilon u(x/\epsilon)] = u'(x/\epsilon)$ is of order unity, so a nonlinear term like uu_x is of the same order as u. This same idea was used by Chen & Keller (1973) to derive a K.d.V.-like equation for weak waves in water of non-uniform depth.

The outcome of this theory is that waves travel along the rays of ordinary geometrical optics, which are independent of the wave amplitude. However, the amplitude satisfies a nonlinear equation along these rays. Thus the theory combines some of the simplicity of the usual ray methods with some of the features of nonlinear propagation, such as waveform distortion, harmonic generation, and shock formation. It has been used by Hunter & Keller (1984) to treat weak shock diffraction by a wedge, so it provides a basis for developing a scattering theory for weakly nonlinear waves. In addition, J. Hunter, A. Majda and R. Rosales, in unpublished work, have extended the theory to describe the resonant interactions of systems of such waves.

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Discussion

A. D. D. Craik (*Department of Applied Mathematics, University of St Andrews, U.K.*). Professor Keller has described very interesting experiments of Wehausen, which show that a train of 'solitary waves' sometimes propagate ahead of a body moving uniformly through shallow water. I am reminded of a feature recently observed in some satellite photographs of cloud structure over Jan Mayen Island, and shown to me by Professor R. Scorer. In addition to the familiar 'ship-wave pattern' of lee waves behind the island, there is sometimes a long dark band stretching from the island in a direction roughly perpendicular to the wind. I wonder whether this too could be a 'solitary wave'.

J. B. Keller. These lee waves are internal waves in a stratified fluid, and there do exist solitary internal waves. Such a wave would be stationary if the wind velocity was just opposite that of the soliton. Therefore these velocities would have to be examined to see if they could fulfil this condition.

J. T. Stuart, F.R.S. (*Department of Mathematics, Imperial College, London, U.K.*). Beyond the Hopf-bifurcation point for the Fitzhugh–Nagamo equations, it should be possible to derive an amplitude equation or time-evolution equation by means of weakly nonlinear multiple-scaling theory. This would enable a study of the nonlinear time-dependent solution in that neighbourhood.

P. L. Christiansen (*Laboratory of Applied Mathematical Physics, The Technical University of Denmark, Denmark*). Many soliton scattering problems in connection with the perturbed sine–Gordon equation have been solved and used to explain the dynamical behaviour of Josephson junction oscillators observed experimentally. Often the computational results are very well predicted by soliton perturbation theory. Instabilities of linear modes give rise to creation of soliton dynamic states. Modulation theory is used to account for the switching between different branches of the current–voltage characteristic of the oscillator.

R. K. Bullough (*Department of Mathematics, UMIST, Manchester, U.K.*). I would like to comment on the suggested use of Whitham's averaged Lagrangian method. In collaboration with J. A. Armstrong (I.B.M., Yorktown Heights, New York), and following pioneering work by J. C. Eilbeck in this Department in 1970–72, we made a substantial application of the method in Manchester in 1975–78 (cf. Jack 1978; Bullough *et al.* 1979). The application was to the problem of self-induced transparency: the propagation of nanosecond or shorter resonant, or almost resonant, optical pulses in a nonlinear medium. The pulse envelopes finally prove to be described by integrable equations that have multisoliton solutions and the pulses, which can be observed (and have been observed by photodiodes), are fine examples of true solitons.

The nonlinear medium is first described as good physics through the so-called coupled Bloch–Maxwell equations. These are not exactly integrable, but a multiple scales analysis shows that a hierarchy of physically significant integrable equations is contained within them, namely the 'reduced Maxwell–Bloch', the 'self-induced transparency', and the sine–Gordon equations (Eilbeck *et al.* 1973). The multiple scales analysis is equivalent to what physicists call the slowly varying envelope and phase approximation (s.v.e.p.a.) and both use the point that pulse envelopes develop on a 10^{-9} s timescale, but the resonant carrier wave (the light) oscillates at 10^{15} Hz. This is exactly the 'two timing' situation envisaged by Whitham, and because the results should be integrable equations with well known soliton solutions (these are observed after all!) the system is ideal for a test of Whitham's averaged Lagrangian technique.

The fact is that we found this technique very difficult to apply successfully. The simplest application of the technique (namely substitution of the travelling wave solutions into the Lagrangian density and averaging over one cycle of the periodic travelling wave with amplitude and frequency assumed constant over the interval) can be justified only as a zero-order approximation in a formal perturbation scheme. The results obtained at this order were in good agreement *for small times* with experiments on the self-steepening of pulses (Grischkowsky *et al.* 1973), but only *sufficiently far from resonance*. They agreed with the so-called 'adiabatic following approximation'; but in the longer term shocks developed (Armstrong 1975). These shocks are entirely due to the approximation: there are no shocks in self-induced transparency where nonlinearity is always balanced by dispersion.

To handle the technique correctly, dispersive terms must be introduced, and we followed Yuen & Lake (1975) first of all (also see Chu & Mei (1970) for a heuristic argument in this connection) to introduce terms into the averaged Lagrangian $\bar{L}$, which experts will recognize in the form $\frac{1}{2}(G_{\omega\omega} A_t^2 + 2G_{\omega k} A_t A_z + G_{kk} A_z^2)$ (the field variable is the vector potential with travelling *wave amplitude* A and $A_t \equiv \partial A/\partial t$, etc.). However, reference to adiabatic following showed that this is not enough and further terms like $K(\Delta\omega)^{-3}(A_{tt}A - 2A_t^2)$ (K is a constant) also had to be inserted so that the 'wave action' equation could admit the necessary hyperbolic secant (soliton) solution. But these additions to $\bar{L}$ also changed the dispersion relation and 'conservation of waves' condition and thus were unacceptable alone. Note in any case that, since $\Delta\omega$ is the frequency offset from resonance, these terms rapidly become 'zero order' as resonance is approached. In fact to handle the resonance one must work at all orders of perturbation theory of course. For circularly polarized light the travelling wave solutions are trigonometric and it is actually possible to carry out this programme completely and rather easily. However, for plane polarized light the travelling waves are Jacobian elliptic functions and only a low-order theory was achieved. However, replacing the elliptic functions by trigonometric functions led us to results in agreement with multiple scales and the s.v.e.p.a. We were able to extend the technique to oppositely directed colliding pulses by a two-phase Whitham-type analysis and agreement with the s.v.e.p.a. was again achieved at low order. But to study the collision of oppositely directed pulses in a collaboration with experiment (see my comment following the paper by Dr Mollenauer (this symposium)) we found ourselves obliged to work to all orders again and could only do this through multiple scales (i.e. through the s.v.e.p.a.).

In summary, the method of multiple scales (the s.v.e.p.a.) always works (if the physics is right), but averaged Lagrangian techniques apparently designed for exactly the same physical two-timing situation are extremely difficult to use near a resonance. We had to use the s.v.e.p.a.

to guide the averaged Lagrangian through the resonance regions and this could actually be done only because of special features associated with the trigonometrical travelling wave solutions associated with circularly polarized light. The moral seems to be that averaged Lagrangian techniques can only be used with caution and with profound understanding.

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